

Non-abelian expansion of S-matrix elements and non-abelian tachyon DBI action

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ABSTRACT

We apply the prescription proposed in hep-th/0307197 for non-abelian expansion of S-matrix elements, to the S-matrix element of four tachyons and one gauge field in superstring theory. We show that the leading order terms of the expansion are in perfect agreement with the non-abelian generalization of the tachyon DBI action in which the tachyon potential is $V(T) = 1 + \pi\alpha' m^2 T^2 + \frac{1}{2!}(\pi\alpha' m^2 T^2)^2 + \dots$ where $m^2 = -1/(2\alpha')$ is the mass of tachyon. This calculation fixes the coefficient of four-tachyon couplings without on-shell ambiguity.

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1 Introduction

Decay of unstable branes is an interesting process which might shed new light in understanding properties of string theory in time-dependent backgrounds [1]-[32]. In particular, by studying the unstable branes in the boundary conformal field theory (BCFT), Sen has shown that the end of tachyon rolling in this theory is a tachyon matter with zero pressure and non-zero energy density [3]. These results can be reproduced in field theory using the tachyon DBI action [33, 34]. See [35] for review.

The tachyon DBI action was originally proposed as an action which is consistent with the non-commutative expansion of the S-matrix elements involving open string tachyons [33]. The example considered in [33] was the S-matrix element of one graviton and two tachyons in the presence of background B-flux. Using the fact that the field theory on the world-volume of D-brane with B-flux is a non-commutative theory [36]-[40], one naturally expects there should be an expansion for the S-matrix element whose first leading order term which is a massless pole is reproduced by the non-commutative kinetic term of the tachyon action. Such expansion which is not the usual α' -expansion has been found for the above S-matrix element in [33]. It is an expansion around $s \rightarrow 0$ where the Mandelstam variable is $s = -1/2 - \alpha' k_1 \cdot k_2$ where k_1, k_2 are the momenta of the tachyons. It has been shown while the first leading order term of the expansion is exactly reproduced by the non-commutative kinetic term, the next order terms which are contact terms, indicates that the kinetic term of tachyon can not be in the standard form. It should appear, in fact, in the tachyon DBI form [33]¹.

When there are N coincident D-branes, the $U(1)$ gauge symmetry of an individual D-brane is enhanced to the non-abelian $U(N)$ symmetry [43]. In this case, one expects there should be an expansion for any string theory S-matrix element whose first leading order terms are reproduced by the non-abelian gauge invariant kinetic term of the tachyon action. This expansion has been called in [45] the non-abelian expansion of the S-matrix elements. As an example for this case, the non-abelian expansion of the S-matrix element of four tachyons has been found in [45]. This expansion is not the usual α' -expansion. Like the S-matrix element of four massless transverse scalars, the contact terms of the expansion are ordered in terms of $\zeta(2), \zeta(3), \dots$. However, at each $\zeta(n)$ -order, there are terms at different α' order. It has been shown that while the first leading order terms of the expansion which are massless poles are reproduced by the non-abelian kinetic term, the next leading order terms which are contact terms are exactly reproduced by the non-abelian generalization of tachyon DBI action in which the tachyon potential is $V(T) = 1 + \pi\alpha'm^2T^2 + \frac{1}{2!}(\pi\alpha'm^2T^2)^2 + \dots$ where m^2 is mass of tachyon. Since these couplings are fixed by comparing the contact terms of the S-matrix element, there is the on-shell ambiguity in calculating these terms, *e.g.*, the coefficient of T^4 in the tachyon potential has

¹Throughout the paper, we are appealing to the specific meaning of the tachyon action as a generating functional for producing the leading terms of the string theory S-matrix elements.

the ambiguity that one can not distinguish between, say, T^4 and $T^3\partial^2T$. Both have the same contribution to the S-matrix element of four tachyons. This on-shell ambiguity can be fixed by studying the S-matrix elements in which one of the tachyon appears as off-shell in the Feynman amplitude, *e.g.*, the S-matrix element of four tachyons and one gauge field.

The non-commutative/non-abelian expansion is not α' -expansion, so one may conclude that the non-leading terms of the expansion are not related to higher derivative correction to tachyon DBI action². However, using the fact that for on-shell tachyon $\alpha'\partial_a\partial^aT \sim T$, the higher derivatives of tachyon action at a fixed α' order may produce contact terms at various α' order in the S-matrix element. Moreover, if a tachyon potential multiplies a higher derivative term, then higher derivative terms at a fixed number of derivative of tachyon *e.g.*, $\partial\partial T$ -order, produce different α' order in the S-matrix element, *e.g.*, $\alpha'^2T^2\partial_a\partial_bT\partial^a\partial^bT$ and $\alpha'^3\partial_cT\partial^cT\partial_a\partial_bT\partial^a\partial^bT$ are both at second-order derivative of tachyon, however they produce different α' order terms in the S-matrix element of four tachyons. Hence, one can not argue that the non-leading terms of the expansion are not related to the higher derivative correction to tachyon DBI action either. Using the fact that the S-matrix method can not fix the coefficients of all gauge invariant structures uniquely, *e.g.*, there is a field redefinition freedom [41, 42], one may correspond the non-leading terms of the non-abelian expansion to the second-, third- and higher-derivative corrections to the non-abelian tachyon DBI action.

In superstring theory, the non-abelian expansion of a S-matrix element can be found easily [46]. The S-matrix elements of odd number of tachyon vertex operators are zero, and expansion of a S-matrix element of even number of tachyons can be found as the following: Both tachyon and massless transverse scalars transform in the adjoint representation of $U(N)$ group, hence, they both have identical non-abelian kinetic terms. The Feynman amplitudes resulting from the kinetic terms are then similar in both cases. Therefore, the non-abelian expansion of a S-matrix element of even number of tachyons and the non-abelian expansion of the S-matrix element in which the tachyons are replaced by the transvers scalars should be similar. On the other hand, the non-abelian expansion of a S-matrix element of even number of massless scalars is known, *i.e.*, sending all Mandelstam variables to zero. Using similar steps for the S-matrix element of tachyons, one would find the non-abelian expansion of the tachyon amplitude [46]. In the bosonic theory, however, the S-matrix elements of odd number of tachyons are non-zero and the S-matrix elements in which the tachyons are replaced by massless scalar fields are zero. Hence, one can not use the above prescription to find the non-abelian expansion of the S-matrix element of odd number of tachyons.

An observation made in [48] is that the S-matrix elements of four tachyons and the S-matrix element in which the tachyons are replaced by the massless scalars can be written

²See [49], for discussion on validity of tachyon DBI action as effective theory of non-BPS D-brane for spacial slowly varying tachyon field in tachyon rolling background in BSCT. There is a natural α' expansion for the tachyon S-matrix elements here, however, expansion is in terms of spatial momentum of tachyons.

in a universal form. We speculate that this property holds for any S-matrix element of even number of tachyons. This indicates that the on-shell mass of tachyons/scalars does not appear in the S-matrix element in the universal form. In order to reproduce the leading terms of the expansion by a tachyon action, however, one must use the relations in which the Mandelstam variables are satisfied, *e.g.*, (6). Using the fact that these relations involve the mass of tachyon, one realizes that the tachyon action should include the tachyon mass. One may take the fact that the tachyon mass does not appear in the string theory side as an evidence that the mass in field theory side is arbitrary. However, this field theory is valid only for those open string vertex operators whose S-matrix elements can be written in the universal form, *i.e.*, the open string tachyon of N non-BPS D_p -branes³.

In this paper, we would like to calculate the S-matrix element of four tachyons and one gauge field, and find the non-abelian expansion of the amplitude. One can find this expansion in two different ways. 1-Comparing the amplitude with the amplitude of four transverse scalars and one gauge field. Non-abelian expansion of the latter amplitude can be found by sending all Mandelstam variables to zero. Similar expansion for the S-matrix element of tachyons gives the non-abelian expansion. 2-Using the on-shell constraints that the Mandelstam variables satisfy, *i.e.*, (6), one may write the S-matrix element of tachyons in the universal form. Then send all Mandelstam variables to zero. In the present paper, we will use the first approach to find the non-abelian expansion. In sect.2, we will calculate the S-matrix element of four tachyons and one gauge field. In order to find the non-abelian expansion of the S-matrix element, we shall calculate the S-matrix element of four scalars and one gauge field in sect.3.1, and then find the non-abelian expansion of the tachyon amplitude. In sect.4, we shall show that while the first leading order terms of the expansion which are tachyon and massless poles are reproduced by non-abelian kinetic term of tachyon, the next leading order terms which are tachyon poles, massless poles and contact terms are fully consistent with non-abelian tachyon DBI action. This calculation fixes the four-tachyon coupling in the tachyon DBI action without on-shell ambiguity.

2 Tachyon amplitude in superstring theory

Using the world-sheet conformal field theory technique [50], one can evaluate a 5-point function by evaluating the correlation function of their corresponding vertex operators. Performing the correlations, one finds that the integrand has $SL(2, R)$ symmetry. Then one should fix this symmetry by fixing position of three vertices in the real line. Different fixing

³In [45], it was shown that the S-matrix element of four massive scalars with vertex operator $\lambda \int dx (\zeta_i \partial^n X^i) e^{ik \cdot X}$ where $k^2 = -(n-1)/\alpha'$, can be written in the universal form. Then it was concluded that various coupling of these scalars are given by the tachyon DBI action in which the mass of tachyon in the potential $V(T) = e^{\pi \alpha' m^2 T^2}$ is replaced by $m^2 = (n-1)/\alpha'$. However, the above vertex is not a primary operator. The primary massive vertex operators can not be written in the universal form. Hence, the tachyon DBI action can be used only for the tachyon vertex operator of N non-BPS D_p -branes.

of these positions give different ordering of the five vertices in the boundary of the world-sheet. One should add all non-cyclic permutation of the vertices to get the correct scattering amplitude. For the purpose of comparing string theory S-matrix element with field theory S-matrix element, it is enough to consider only a special ordering, *i.e.*, $(x_1 = 0, x_2, x_3, x_4 = 1, x_5 = \infty)$ where the label 5 refers to the gauge field in the amplitude. With this ordering, one finds that the S-matrix element has the Chan-Paton factor $\text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5)$. After fixing the $SL(2, R)$, one ends up with one double integral which can be performed. The result is a multiple of two Beta functions and one Hypergeometric function [51].

In particular, the S-matrix element of four tachyons and one gauge field of N non-BPS D_p -branes in superstring theory is given by the following correlation function:

$$\mathcal{A}^{TTTTA} \sim \int dx_1 dx_2 dx_3 dx_4 dx_5 \langle V_0^T V_0^T V_0^T V_{-1}^T V_{-1}^A \rangle \quad (1)$$

where the vertex operators for tachyons and gauge field are given by

$$\begin{aligned} V_0^T &= \zeta^\alpha (2ik \cdot \psi) e^{2ikX} \Lambda^\alpha \\ V_{-1}^T &= \zeta^\alpha e^{2ikX} e^{-\Phi} \Lambda^\alpha \\ V_{-1}^A &= \xi_a^\alpha \psi^a e^{2ikX} e^{-\Phi} \Lambda^\alpha \end{aligned}$$

where $\alpha = 1, 2, \dots, N^2$ and N is the number of D_p -branes. ζ is polarization of tachyon which specifies one of the N^2 open string tachyons existed in the D_p -branes system. Similarly, ξ_a^α is the polarization of gauge field where the index a specifies the component of gauge field in the world volume of D_p -branes. The matrices Λ^α are the complete bases in terms of which the Chan-Paton matrix can be expanded, *i.e.*, $\lambda = \zeta^\alpha \Lambda^\alpha$ for tachyon, and $\xi_a \lambda = \xi_a^\alpha \Lambda^\alpha$ for gauge field. In the above vertices, k is the world volume momentum of the open string states. The on-shell condition for tachyon is $k^2 = 1/(2\alpha')$, and for gauge field is $\xi_a k^a = 0$ and $k^2 = 0$. X^a , ψ^a and Φ are the usual world-sheet fields of the fermionic string (see [50] for details). The two dimensional world sheet fields have the standard propagators:

$$\begin{aligned} \langle X^\mu(z) X^\nu(w) \rangle &= -\eta^{\mu\nu} \log(z - w) \\ \langle \psi^\mu(z) \psi^\nu(w) \rangle &= -\frac{\eta^{\mu\nu}}{z - w} \\ \langle \Phi(z) \Phi(w) \rangle &= -\log(z - w) \end{aligned} \quad (2)$$

Using the conformal field theory technique, one can evaluate the correlation functions and show that integrand has the $SL(2, R)$ symmetry. Fixing this symmetry as $(x_1 = 0, x_2, x_3, x_4 = 1, x_5 = \infty)$, one finds the final result as

$$\begin{aligned} \mathcal{A}^{TTTTA} &= -4i\alpha' T_p \text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5) \\ &\times \left((-s_{23} - 1)a' k_1 \cdot \xi_5 + (s_{13} + 1)b' k_2 \cdot \xi_5 + (-s_{12} - 1)c' k_3 \cdot \xi_5 \right) \end{aligned} \quad (3)$$

where a' , b' , and c' are the double integrals that left over after fixing the position of three vertex operators. These integrals are discussed in the appendix A. We have also normalized the amplitude by $-4i\alpha'T_p$. To compare the above S-matrix element with the corresponding amplitude in field theory, one needs only to keep the first term, *i.e.*, $(a'k_1 \cdot \xi_5)$ term. In terms of the Beta and Hypergeometric functions, this term is

$$\begin{aligned} \mathcal{A}^{TTTTA} = & -4i\alpha'T_p \text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5) k_1 \cdot \xi_5 \\ & \left((-s_{23} - 1) \beta(-s_{12}, -s_{23} - 1) \beta(-s_{45} - \frac{1}{2}, -s_{34}) \right. \\ & \left. \times {}_3F_2(-s_{12}, -s_{45} - \frac{1}{2}, s_{15} - s_{23} - s_{34} - \frac{1}{2}; -s_{12} - s_{23} - 1, -s_{34} - s_{45} - \frac{1}{2}; 1) \right) \end{aligned} \quad (4)$$

where the definition of kinematic factors s_{ij} is

$$s_{ij} = -\alpha'(k_i + k_j)^2 \quad (5)$$

The number of independent kinematic factors in the scattering amplitude of n states is $\frac{n}{2}(n-3)$ [52]. In the present case, there are 5 independent kinematic factors. In writing the above amplitude, we have chosen them as $s_{12}, s_{23}, s_{34}, s_{45}, s_{15}$. Using conservation of momentums and the on-shell conditions $k_1^2 = k_2^2 = k_3^2 = k_4^2 = 1/(2\alpha')$, $k_5^2 = 0$, one finds that the other kinematic factors $s_{13}, s_{14}, s_{24}, s_{25}, s_{35}$ can be written in terms of independent ones as

$$\begin{aligned} s_{13} &= s_{45} - s_{12} - s_{23} - \frac{3}{2} \\ s_{14} &= s_{23} - s_{15} - s_{45} - \frac{1}{2} \\ s_{24} &= s_{15} - s_{23} - s_{34} - \frac{3}{2} \\ s_{25} &= s_{34} - s_{12} - s_{15} - 1 \\ s_{35} &= s_{12} - s_{45} - s_{34} - 1 \end{aligned} \quad (6)$$

One can also show that the Mandelstam variables satisfied the following relation:

$$\sum_{i < j} s_{ij} = -6 \quad (7)$$

For later purposes, it is convenient to absorb the factor $(s_{23} + 1)$ in (4) into the first Beta function, *i.e.*,

$$\begin{aligned} \mathcal{A}^{TTTTA} = & -4i\alpha'T_p \text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5) k_1 \cdot \xi_5 \\ & \left(\frac{\Gamma(-s_{12})\Gamma(-s_{23})}{\Gamma(-1 - s_{12} - s_{23})} \beta(-s_{45} - \frac{1}{2}, -s_{34}) \right. \\ & \left. \times {}_3F_2(-s_{12}, -s_{45} - \frac{1}{2}, s_{15} - s_{23} - s_{34} - \frac{1}{2}; -s_{12} - s_{23} - 1, -s_{34} - s_{45} - \frac{1}{2}; 1) \right) \end{aligned} \quad (8)$$

From poles of the Beta function in (4), one realizes that the amplitude has tachyon, massless and infinite tower of massive poles. Some of the tachyon and massless poles should be reproduced, in field theory, by non-abelian kinetic term. It is completely nontrivial to find an expansion for Beta and Hypergeometric functions which keeps only the tachyon and massless poles which are reproduced by non-abelian kinetic terms and decouples all other poles. Obviously, because of the tachyon pole, this would not be the usual α' expansion in which tachyon and massive modes are decoupled. However, as we argued in the introduction section, this expansion may lead to an effective action for tachyon and massless fields when second and higher derivative of tachyon is small in compare with the string scale. In the next section we shall find such expansion by comparing it with the S-matrix element of four transverse scalars and one gauge field.

3 Non-abelian Expansion of Tachyon Amplitude

Using the principle that the tachyon action should have the $U(N)$ symmetry, one can find a natural expansion for the tachyon amplitude, *i.e.*, an expansion whose first leading order terms are reproduced by non-abelian kinetic terms. To find this expansion, we note that the tachyon and transverse scalar field of D_8 -brane transform as the adjoint representation of $U(N)$ group, hence, they both have identical non-abelian kinetic term. The Feynman diagrams resulting from the kinetic term are then identical, *i.e.*, the Feynman diagrams of scalar field can be converted to the Feynman diagram of tachyon by replacing each scalar line with the tachyon line (see Fig.1). The Feynman amplitudes are also "similar", *i.e.*, replacing the propagator of scalars with the propagator of tachyon, one can convert the scalar amplitude to the tachyon amplitude. On the other hand, the natural expansion of the S-matrix element of the scalars whose first leading order terms are reproduced by non-abelian kinetic term is known, *i.e.*, sending all Mandelstam variables to zero. Using "similar" expansion for the tachyon amplitude, one can find the desired non-abelian expansion for tachyon amplitudes. Hence, to find the non-abelian expansion of the tachyon amplitude, we need to recall the non-abelian/ α' expansion of the S-matrix element of four scalars and one gauge field.

3.1 Massless amplitude

The S-matrix element of four transverse scalars and one vector of N D_8 -beanes in the superstring theory can be read from the S-matrix element of five gauge fields [51, 56]. However, our notation for open string momenta and s_{ij} is different from those references. In our notation, the amplitude is given by

$$\mathcal{A}^{\phi\phi\phi\phi A} \sim \int dx_1 dx_2 dx_3 dx_4 dx_5 \langle V_0^\phi V_0^\phi V_0^\phi V_{-1}^\phi V_{-1}^A \rangle \quad (9)$$

where the vertex operators are given as

$$\begin{aligned} V_0^\phi &= (\partial X^9 + 2ik \cdot \psi \psi^9) e^{2ikX} \lambda \\ V_{-1}^\phi &= \psi^9 e^{2ikX} e^{-\Phi} \lambda \\ V_{-1}^A &= \xi_a \psi^a e^{2ikX} e^{-\Phi} \lambda \end{aligned}$$

where ξ^a is the polarization of gauge field. Performing the correlators in (9), one finds the following final result:

$$\begin{aligned} \mathcal{A}^{\phi\phi\phi\phi A} &= -4i\alpha' T_8 \text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5) \left[k_1 \cdot \xi_5 \left(-s_{23} a_1 + s_{23} a_2 - (s_{23} + 1) a_3 \right) \right. \\ &\quad \left. + k_2 \cdot \xi_5 \left(s_{13} b_1 - (s_{13} + 1) b_2 + s_{13} b_3 \right) + k_3 \cdot \xi_5 \left(- (s_{12} + 1) c_1 + s_{12} c_2 - s_{12} c_3 \right) \right] \end{aligned} \quad (10)$$

where the coefficients a_i, b_i, c_i for $i = 1, 2, 3$, which are the double integrals are discussed in the appendix A. We have also normalized the amplitude by the factor $-4i\alpha' T_8$. The definition of the kinematic factors s_{ij} is the same as before, *i.e.*, (5). However, because of the on-shell condition $k_1^2 = k_2^2 = k_3^2 = k_4^2 = k_5^2 = 0$ in the present case, one finds the following relation between s_{ij} :

$$\begin{aligned} s_{13} &= s_{45} - s_{12} - s_{23} \\ s_{14} &= s_{23} - s_{15} - s_{45} \\ s_{24} &= s_{15} - s_{23} - s_{34} \\ s_{25} &= s_{34} - s_{12} - s_{15} \\ s_{35} &= s_{12} - s_{45} - s_{34} \end{aligned}$$

and $\sum_{i < j} s_{ij} = 0$. Again, we consider only $k_1 \cdot \xi_5$ terms. Using the above relations, one can arrange the result as

$$\begin{aligned} \mathcal{A}^{\phi\phi\phi\phi A} &= -4i\alpha' T_8 \text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5) k_1 \cdot \xi_5 \\ &\quad \left(-s_{23} \beta(-s_{12}, -s_{23}) \beta(-s_{45}, -s_{34}) \right. \\ &\quad \times {}_3F_2(-s_{12}, -s_{45}, s_{15} - s_{23} - s_{34}; -s_{12} - s_{23}, -s_{34} - s_{45}; 1) \\ &\quad + s_{23} \beta(1 - s_{12}, -s_{23}) \beta(-s_{45}, 1 - s_{34}) \\ &\quad \times {}_3F_2(1 - s_{12}, -s_{45}, 1 + s_{15} - s_{23} - s_{34}; 1 - s_{12} - s_{23}, 1 - s_{34} - s_{45}; 1) \\ &\quad - (s_{23} + 1) \beta(1 - s_{12}, -1 - s_{23}) \beta(-s_{45}, 1 - s_{34}) \\ &\quad \left. \times {}_3F_2(1 - s_{12}, -s_{45}, s_{15} - s_{23} - s_{34}; -s_{12} - s_{23}, 1 - s_{34} - s_{45}; 1) \right) \end{aligned}$$

One can α' expand the Beta and Hypergeometric functions (see appendix A) to find the non-abelian/ α' expansion of the above scattering amplitude, and compare the results with the low energy field theory. We have done this calculation and found that the leading order terms of the expansion are in perfect agreement with the non-abelian DBI action [53].

Similar calculation for the S-matrix element of five gauge fields has been done in [51, 56]. However, our main purpose behind calculating the above scalar amplitude is to find a guide for expanding the Beta and Hypergeometric functions in the tachyon amplitude (4). To this end, we first rewrite the above amplitude as

$$\begin{aligned}
\mathcal{A}^{\phi\phi\phi\phi A} = & -4i\alpha'T_8\text{Tr}(\lambda^1\lambda^2\lambda^3\lambda^4\lambda^5)k_1\cdot\xi_5 \\
& \left(\frac{\Gamma(-s_{12})\Gamma(1-s_{23})}{\Gamma(-s_{12}-s_{23})}\beta(-s_{45},-s_{34}) \right. \\
& \times {}_3F_2(-s_{12},-s_{45},s_{15}-s_{23}-s_{34};-s_{12}-s_{23},-s_{34}-s_{45};1) \\
& - \frac{\Gamma(1-s_{12})\Gamma(1-s_{23})}{\Gamma(1-s_{12}-s_{23})}\beta(-s_{45},1-s_{34}) \\
& \times {}_3F_2(1-s_{12},-s_{45},1+s_{15}-s_{23}-s_{34};1-s_{12}-s_{23},1-s_{34}-s_{45};1) \\
& + \frac{\Gamma(1-s_{12})\Gamma(-s_{23})}{\Gamma(-s_{12}-s_{23})}\beta(-s_{45},1-s_{34}) \\
& \left. \times {}_3F_2(1-s_{12},-s_{45},s_{15}-s_{23}-s_{34};-s_{12}-s_{23},1-s_{34}-s_{45};1) \right)
\end{aligned}$$

Note that each term in the above amplitude has the same structure as the tachyon amplitude (8). The non-abelian expansion of the above scalar amplitude is obviously expansion around $s_{ij} \rightarrow 0$. This sends the argument of the above functions to different points, *i.e.*, the Gamma, Beta and Hypergeometric functions in the first, second, and third term must be expanded around, respectively,

$$\begin{aligned}
\text{first term} & \rightarrow \frac{\Gamma(0)\Gamma(1)}{\Gamma(0)}\beta(0,0){}_3F_2(0,0,0;0,0;1) \\
\text{second term} & \rightarrow -\frac{\Gamma(1)\Gamma(1)}{\Gamma(1)}\beta(0,1){}_3F_2(1,0,1;1,1;1) \\
\text{third term} & \rightarrow \frac{\Gamma(1)\Gamma(0)}{\Gamma(0)}\beta(0,1){}_3F_2(1,0,0;0,1;1)
\end{aligned} \tag{11}$$

Our claim is that to produce the non-abelian expansion for the tachyon amplitude (8), one should expand the amplitude around the same points, *i.e.*, the arguments of Gamma, Beta and Hypergeometric functions in the tachyon amplitude (8) should be sent to the same points. For example, in order to send the arguments of Gamma, Beta and Hypergeometric functions in (8) to $\frac{\Gamma(0)\Gamma(1)}{\Gamma(0)}\beta(0,0){}_3F_2(0,0,0;0,0;1)$, one should send $s_{23} \rightarrow -1$, $s_{45}, s_{15} \rightarrow -\frac{1}{2}$, $s_{12}, s_{34} \rightarrow 0$. Similarly for other points. Hence, our proposal for non-abelian expansion of the tachyon amplitude is the following:

$$\begin{aligned}
\mathcal{A}^{TTTTA} = & \left(\lim_{s_{23} \rightarrow -1, s_{45}, s_{15} \rightarrow -\frac{1}{2}, s_{12}, s_{34} \rightarrow 0} \mathcal{A}^{TTTTA} \right) \\
& - \left(\lim_{s_{45}, s_{15} \rightarrow -\frac{1}{2}, s_{12}, s_{23}, s_{34} \rightarrow -1} \mathcal{A}^{TTTTA} \right) + \left(\lim_{s_{45}, s_{15} \rightarrow -\frac{1}{2}, s_{12}, s_{34} \rightarrow -1, s_{23} \rightarrow 0} \mathcal{A}^{TTTTA} \right)
\end{aligned} \tag{12}$$

Note that if one ignores the limit signs, the right hand side becomes \mathcal{A}^{TTTTA} . Moreover, as we already mentioned around eq.(6), we have chosen the independent Mandelstam variables to be $s_{12}, s_{23}, s_{34}, s_{45}, s_{15}$, hence, the above is a linear combination of three on-shell limits. Now, one can expand the Beta and Hypergeometric functions in (4) around the above points (see appendix A). The result is the following:

$$\begin{aligned} \mathcal{A}^{TTTTA} = & 4i\alpha' T_p \text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5) \xi_5 \cdot k_1 \quad (13) \\ & \left[\left(\frac{s_{23}}{(s_{45} + \frac{1}{2})s_{12}} + \frac{s_{23}}{(s_{15} + \frac{1}{2})s_{34}} + \frac{s_{23}}{s_{12}s_{34}} + \frac{s_{12}}{s_{23}(s_{45} + \frac{1}{2})} + \frac{s_{34}}{s_{23}(s_{15} + \frac{1}{2})} \right. \right. \\ & + \frac{1}{(s_{45} + \frac{1}{2})s_{12}} + \frac{1}{(s_{15} + \frac{1}{2})s_{34}} + \frac{1}{s_{12}s_{34}} + \frac{1}{s_{23}(s_{45} + \frac{1}{2})} + \frac{1}{s_{23}(s_{15} + \frac{1}{2})} \\ & + \left. \frac{1}{(s_{15} + \frac{1}{2})} + \frac{1}{(s_{45} + \frac{1}{2})} - \frac{1}{s_{23}} \right) \\ & - \zeta(2) \left(\frac{s_{12}^2}{(s_{45} + \frac{1}{2})} + \frac{s_{23}^2}{(s_{45} + \frac{1}{2})} + \frac{s_{12}s_{23}}{(s_{45} + \frac{1}{2})} + \frac{2s_{12}}{(s_{45} + \frac{1}{2})} + \frac{2s_{23}}{(s_{45} + \frac{1}{2})} + \frac{1}{(s_{45} + \frac{1}{2})} \right. \\ & + \frac{s_{23}^2}{(s_{15} + \frac{1}{2})} + \frac{s_{34}^2}{(s_{15} + \frac{1}{2})} + \frac{s_{23}s_{34}}{(s_{15} + \frac{1}{2})} + \frac{2s_{23}}{(s_{15} + \frac{1}{2})} + \frac{2s_{34}}{(s_{15} + \frac{1}{2})} + \frac{1}{(s_{15} + \frac{1}{2})} \\ & + \frac{s_{23}s_{12}}{s_{34}} + \frac{s_{23}(s_{15} + \frac{1}{2})}{s_{34}} + \frac{s_{12}}{s_{34}} + \frac{(s_{15} + \frac{1}{2})}{s_{34}} + \frac{s_{23}s_{34}}{s_{12}} + \frac{s_{23}(s_{45} + \frac{1}{2})}{s_{12}} + \frac{s_{34}}{s_{12}} \\ & + \frac{(s_{45} + \frac{1}{2})}{s_{23}} + \frac{s_{12}(s_{15} + \frac{1}{2})}{s_{23}} + \frac{s_{34}(s_{45} + \frac{1}{2})}{s_{23}} + \frac{(s_{45} + \frac{1}{2})}{s_{23}} - \frac{(s_{15} + \frac{1}{2})(s_{45} + \frac{1}{2})}{s_{23}} \\ & + \left. \frac{(s_{15} + \frac{1}{2})}{s_{23}} - s_{12} - s_{34} + s_{15} + s_{45} - 3s_{23} - 3 \right) + \dots \Big] \end{aligned}$$

where dots represent terms that are proportional to $\zeta(3)$, $\zeta(4)$, and so on. This is the non-abelian expansion of the tachyon amplitude (4) we were after. As we anticipated before, the above non-abelian expansion keeps some of the tachyon and massless poles of amplitude (4) and decouples all other poles. Unlike the non-abelian expansion of the S-matrix elements of massless states, the above expansion is not an α' expansion. It has the same $\zeta(n)$ expansion as the massless scalar amplitude. However, unlike the scalar amplitude, the different terms at each $\zeta(n)$ have different α' order. In the field theory side, this indicates that the terms at each $\zeta(n)$ may have different α' order terms. In the next section, we shall show that the terms in the first three lines above are reproduced by non-abelian kinetic terms *i.e.*, the terms in the first line of (17), and the terms proportional to $\zeta(2)$ are reproduced by all terms in (17) which have obviously different α' order.

4 Amplitude in Effective Field Theory

The states that appear as on-shell or off-shell in the above S-matrix element are only tachyon and gauge field. We are interested then in the part of effective field theory of D_p -branes which includes only tachyon and gauge fields. Hence, we are looking for the effective action of D_9 -brane in flat background. The proposal is the non-abelian tachyon DBI action of D_9 -branes which is given by⁴

$$\mathcal{L}^{\text{BI}} = -T_9 \text{STr} \left(V(T) \sqrt{-\det(\eta_{ab} + 2\pi\alpha' F_{ab} + 2\pi\alpha' D_a T D_b T)} \right), \quad (14)$$

where the T_9 is the brane tension. The trace on the non-abelian matrices is the symmetric trace. The non-abelian field strength and covariant derivative of tachyon are, respectively,

$$F^{ab} = \partial^a A^b - \partial^b A^a - i[A^a, A^b], \quad D_a T = \partial_a T - i[A_a, T]$$

The tachyon potential has the following expansion:

$$V(T) = 1 + \pi\alpha' m^2 T^2 + \frac{1}{2}(\pi\alpha' m^2 T^2)^2 + \dots \quad (15)$$

where m^2 is the mass squared of tachyon, *i.e.*, $m^2 = -1/(2\alpha')$. The above expansion is consistent with the potential $V(T) = e^{\pi\alpha' m^2 T^2}$ which is the tachyon potential of BSFT [54]⁵.

In the non-abelian field theory (14), A_a and T are in the adjoint representation of the gauge symmetry $U(N)$, where N is the number of D_9 -branes. That is $A_a = A_a^\alpha \Lambda^\alpha$ and $T = T^\alpha \Lambda^\alpha$ where the hermitian matrices Λ^α are the adjoint representation of the $U(N)$ group. Our conventions for Λ^α are

$$\sum_\alpha \Lambda_{ij}^\alpha \Lambda_{kl}^\alpha = \delta_{il} \delta_{jk}, \quad \text{Tr}(\Lambda^\alpha \Lambda^\beta) = \delta^{\alpha\beta}, \quad [\Lambda^\alpha, \Lambda^\beta] = i f^{\alpha\beta\gamma} \Lambda^\gamma$$

One can express the structure constant in terms of the group generators

$$i f^{\alpha\beta\gamma} = \text{Tr}([\Lambda^\alpha, \Lambda^\beta] \Lambda^\gamma)$$

Using the following expansion, one can expand the square root in the non-abelian action (14) to produce various interacting terms

$$\begin{aligned} \sqrt{-\det(M_0 + M)} &= \sqrt{-\det(M_0)} \left(1 + \frac{1}{2} \text{Tr}(M_0^{-1} M) - \frac{1}{4} \text{Tr}(M_0^{-1} M M_0^{-1} M) \right. \\ &\quad + \frac{1}{8} (\text{Tr}(M_0^{-1} M))^2 + \frac{1}{6} \text{Tr}(M_0^{-1} M M_0^{-1} M M_0^{-1} M) \\ &\quad \left. - \frac{1}{8} (\text{Tr}(M_0^{-1} M))^3 + \dots \right) \end{aligned} \quad (16)$$

⁴One may use the non-abelian DBI action of two non-BPS D-branes to find the effective action of D-brane-anti-D-brane [44].

⁵Expanding the square root, and normalizing tachyon as $\frac{1}{2}\pi T^2 \rightarrow 2T^2$, one finds $S = -T_9 e^{-T^2} (1 + 4\alpha' D_a T D^a T + \dots)$ which is the two derivative truncation of BSFT action proposed in [55].

In (14), M_0 and M are

$$\begin{aligned} M_0 &= \eta_{ab} \\ M &= 2\pi\alpha'(F_{ab} + D_a T D_b T) \end{aligned}$$

The terms of the above expansion which has contribution to the S-matrix element of one gauge field and four tachyons are the following:

$$\begin{aligned} \mathcal{L} &= -T_9(\pi\alpha')\text{Tr}\left(m^2 T^2 + D_a T D^a T - (\pi\alpha')F_{ab}F^{ba}\right) \\ &\quad -T_9(2\pi\alpha')^2 S\text{Tr}\left(\frac{m^4}{8}T^4 + \frac{m^2}{4}T^2 D_a T D^a T - \frac{1}{8}(D_a T D^a T)^2\right) \\ &\quad -T_9\frac{(2\pi\alpha')^3}{2}S\text{Tr}\left(D^a T D_b T F^{bc}F_{ca} - \frac{1}{4}D^a T D_a T F^{bc}F_{cb} - \frac{m^2}{4}T T F^{ab}F_{ba}\right) \end{aligned} \quad (17)$$

Note that the above terms are not ordered in terms of power of α' . However, we shall show that the terms in the first line which we call them kinetic order terms, reproduce the first leading terms in (13), *i.e.*, the terms which are not proportional to $\zeta(n)$, and the terms in the second and third lines in combination with the terms in the first line reproduce the next leading order terms in (13), *i.e.*, the terms which are proportional to $\zeta(2)$.

4.1 Kinetic order terms

The non-abelian kinetic terms in the first line of (17) give the following gauge field and tachyon propagators, respectively,

$$\begin{aligned} G_{\alpha\beta}^{ab}(A) &= \frac{i\delta^{ab}\delta_{\alpha\beta}}{(2\pi\alpha')^2 T_9 (-k^2)} \\ G_{\alpha\beta}(T) &= \frac{i\delta_{\alpha\beta}}{(2\pi\alpha')T_9 (-k^2 - m^2)} \end{aligned} \quad (18)$$

where α, β are the group indices and a, b are the world volume indices, and various three and four field couplings. From these couplings, one finds three Feynman diagrams contributing to the S-matrix element of one gauge field and four tachyons (see fig.1). In these diagrams the wavy lines represent gauge field, and straight lines represent tachyon fields. The label 5 is for gauge field and 1, 2, 3, 4 for tachyons. We will choose special ordering for non-abelian fields such that each diagram produces $\text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5)$ factor for its corresponding amplitude, the same group factor that appears in string theory amplitude (13). Our convention for momenta is so that all momenta in each vertex are inwards. Now we compute the Feynman amplitude corresponding to these graphs. We write the amplitude as

$$\mathcal{A}_k^{TTTTA} = A_k + A'_k + A''_k$$

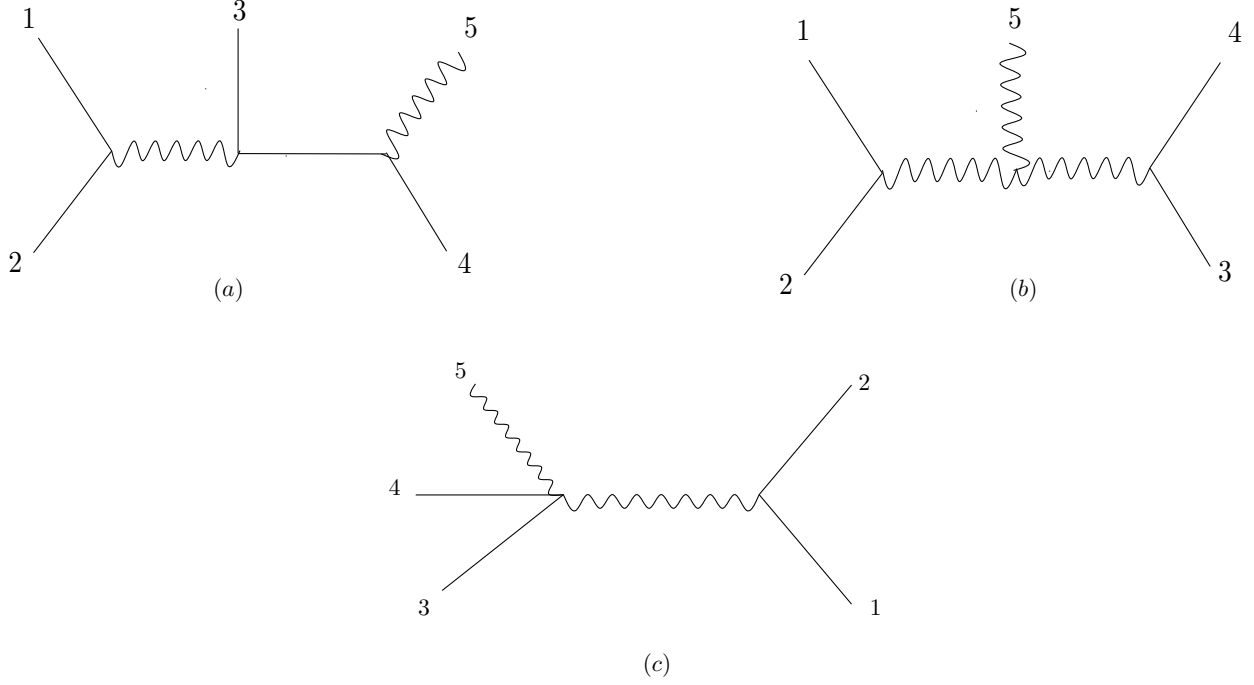


Figure 1: Feynman diagrams resulting from kinetic terms.

where A_k corresponds to diagram (a), A'_k corresponds to diagram (b), and A''_k corresponds to diagram (c).

Using Feynman rules, one can write $A_k(T_1 T_2 T_3 T_4 A_5)$ as the following:

$$A_k(T_1 T_2 T_3 T_4 A_5) = V_\alpha^a(T_1 T_2 A) G_{\alpha\beta}^{ab}(A) V_{\beta\gamma}^b(AT_3 T) G_{\gamma\lambda}(T) V_\lambda(TT_4 A_5) \quad (19)$$

where the propagators are given in (18). The vertices can be written in terms of structure constant or in terms of trace of λ 's. In this section, we use the standard notation and write the vertices of kinetic terms in terms of structure constant. Hence, the above vertices are

$$\begin{aligned} V_\alpha^a(T_1 T_2 A) &= (2\pi\alpha' T_9) \zeta_1^{\alpha_1} \zeta_2^{\alpha_2} (k_1^a - k_2^a) f^{\alpha_1 \alpha_2 \alpha} \\ V_{\beta\gamma}^b(AT_3 T) &= (2\pi\alpha' T_9) \zeta_3^{\alpha_3} (k^b - k_3^b) f^{\beta \alpha_3 \gamma} \\ V_\lambda(TT_4 A_5) &= (2\pi\alpha' T_9) \zeta_4^{\alpha_4} (k - k_4) \cdot \xi_5^{\alpha_5} f^{\lambda \alpha_4 \alpha_5} \end{aligned} \quad (20)$$

where k is momentum of off-shell tachyon (see fig.2). Replacing propagators and vertices in (19), one finds

$$A_k(T_1 T_2 T_3 T_4 A_5) = 2T_9 \zeta_1^{\alpha_1} \zeta_2^{\alpha_2} \zeta_3^{\alpha_3} \zeta_4^{\alpha_4} k_4 \cdot \xi_5^{\alpha_5} \frac{(k_1 - k_2) \cdot (k_4 + k_5 - k_3)}{(k_1 + k_2)^2 ((k_4 + k_5)^2 + m^2)} f^{\alpha_1 \alpha_2 \alpha} f^{\alpha \alpha_3 \beta} f^{\beta \alpha_4 \alpha_5}$$

Note that the amplitude is symmetric under changing $1 \leftrightarrow 2$. The amplitude in terms of s_{ij} can be rewritten as

$$A_k(T_1 T_2 T_3 T_4 A_5) = -(2\alpha' T_9) \zeta_1^{\alpha_1} \zeta_2^{\alpha_2} \zeta_3^{\alpha_3} \zeta_4^{\alpha_4} k_4 \cdot \xi_5^{\alpha_5} f^{\alpha_1 \alpha_2 \alpha} f^{\alpha \alpha_3 \beta} f^{\beta \alpha_4 \alpha_5}$$

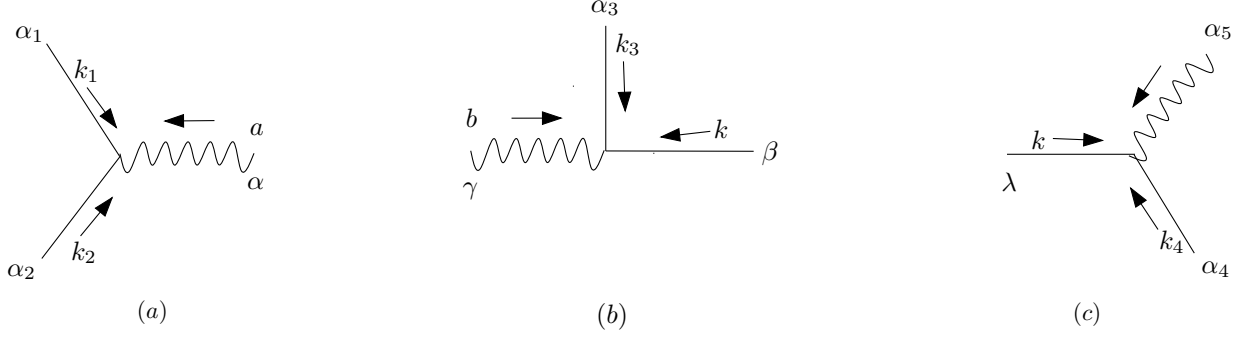


Figure 2: Vertices of two tachyons and one gauge field. a) Two tachyons are on-shell. b) One tachyon is on-shell. c) One tachyon and the gauge field are on-shell.

$$\times \left(-\frac{1}{s_{12}} + \frac{1}{s_{45} + \frac{1}{2}} + \frac{2s_{23}}{(s_{45} + \frac{1}{2})s_{12}} + \frac{2}{(s_{45} + \frac{1}{2})s_{12}} \right)$$

The amplitude in the above form that we shall compare it with (13), is not symmetric under changing $1 \leftrightarrow 2$, however, using the relations in (6), one may rewrite it in a symmetric form. The color factor of the diagram (a) in figure 1 can be expressed in terms of traces of the group generators as the following

$$\begin{aligned} -if^{\alpha_1\alpha_2\alpha} f^{\alpha\alpha_3\beta} f^{\beta\alpha_4\alpha_5} &= -\text{Tr}([\Lambda^{\alpha_1}, \Lambda^{\alpha_2}] \Lambda^{\alpha}) f^{\alpha\alpha_3\beta} f^{\beta\alpha_4\alpha_5} \\ &= \text{Tr}([\Lambda^{\alpha_1}, \Lambda^{\alpha_2}][\Lambda^{\alpha_3}, \Lambda^{\beta}]) if^{\beta\alpha_4\alpha_5} \\ &= \text{Tr}([\Lambda^{\alpha_1}, \Lambda^{\alpha_2}][\Lambda^{\alpha_3}, [\Lambda^{\alpha_4}, \Lambda^{\alpha_5}]]) \end{aligned}$$

In terms of λ 's, one can write

$$-i\zeta_1^{\alpha_1}\zeta_2^{\alpha_2}\zeta_3^{\alpha_3}\zeta_4^{\alpha_4}k_4 \cdot \xi_5 f^{\alpha_1\alpha_2\alpha} f^{\alpha\alpha_3\beta} f^{\beta\alpha_4\alpha_5} = k_4 \cdot \xi_5 \text{Tr}([\lambda^1, \lambda^2][\lambda^3, [\lambda^4, \lambda^5]])$$

This includes among other things our favorite ordering that we had chosen in the string theory side, *i.e.*, $\text{Tr}(\lambda^1\lambda^2\lambda^3\lambda^4\lambda^5)$. Using the fact that in the string theory side (3) there is no term proportional to $k_4 \cdot \xi_5$, one should use conservation of momentum in the field theory side to rewrite $k_4 \cdot \xi_5 = -(k_1 + k_2 + k_3) \cdot \xi_5$. So the above term does has contribution to the $k_1 \cdot \xi_5$ terms.

By changing the labels of fields in the Feynman diagram (a), one can find other diagrams that produce the ordering $\text{Tr}(\lambda^1\lambda^2\lambda^3\lambda^4\lambda^5)$. The color factor of $A_k(T_3T_4T_2T_1A_5)$ is given by

$$\begin{aligned} -i\zeta_1^{\alpha_1}\zeta_2^{\alpha_2}\zeta_3^{\alpha_3}\zeta_4^{\alpha_4}k_1 \cdot \xi_5 f^{\alpha_3\alpha_4\alpha} f^{\alpha\alpha_2\beta} f^{\beta\alpha_1\alpha_5} &= k_1 \cdot \xi_5 \text{Tr}([\lambda^3, \lambda^4][\lambda^2, [\lambda^1, \lambda^5]]) \\ &= k_1 \cdot \xi_5 \text{Tr}(\lambda^1\lambda^2\lambda^3\lambda^4\lambda^5) + \dots \end{aligned}$$

The other part of amplitude $A_k(T_3T_4T_2T_1A_5)$ can be read from $A_k(T_1T_2T_3T_4A_5)$ by changing the numbers 1,2,3,4,5 to the 3,4,2,1,5. That is

$$\begin{aligned} A_k(T_3T_4T_2T_1A_5) &= -(2i\alpha'T_9)k_1 \cdot \xi_5 \text{Tr}(\lambda^1\lambda^2\lambda^3\lambda^4\lambda^5) \\ &\times \left(\frac{1}{s_{34}} - \frac{1}{s_{15} + \frac{1}{2}} - \frac{2s_{23}}{(s_{15} + \frac{1}{2})s_{34}} - \frac{2}{(s_{15} + \frac{1}{2})s_{34}} \right) + \dots \end{aligned}$$

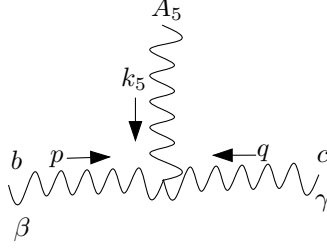


Figure 3: Vertex of one on-shell gauge field and two off-shell gauge fields.

where dots refer to the terms that have ordering other than $\text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5)$. Two other ordering of diagram (a) that produce $\text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5)$ are $A_k(T_2 T_3 T_4 T_1 A_5)$ and $A_k(T_3 T_2 T_1 T_4 A_5)$. One finds these amplitudes from the original amplitude $A_k(T_1 T_2 T_3 T_4 A_5)$, by changing the numbers 1,2,3,4,5 to 2,3,4,1,5 and to 3,2,1,4,5. The results are the following:

$$\begin{aligned}
A_k(T_2 T_3 T_4 T_1 A_5) &= (2i\alpha' T_9) k_1 \cdot \xi_5 \text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5) \\
&\quad \times \left(-\frac{1}{s_{23}} + \frac{1}{s_{15} + \frac{1}{2}} + \frac{2s_{34}}{(s_{15} + \frac{1}{2})s_{23}} + \frac{2}{(s_{15} + \frac{1}{2})s_{23}} \right) + \dots \\
A_k(T_3 T_2 T_1 T_4 A_5) &= -(2i\alpha' T_9) k_4 \cdot \xi_5 \text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5) \\
&\quad \times \left(-\frac{1}{s_{23}} + \frac{1}{s_{45} + \frac{1}{2}} + \frac{2s_{12}}{(s_{45} + \frac{1}{2})s_{23}} + \frac{2}{(s_{45} + \frac{1}{2})s_{23}} \right) + \dots
\end{aligned}$$

Note that in amplitude $A_k(T_3 T_2 T_1 T_4 A_5)$ one should use conservation of momentum to produce the factor $k_1 \cdot \xi_5$ in which we are interested in the string theory side.

Now consider the Feynman diagram (b) in the fig.1. The amplitude is given by:

$$A'_k(T_1 T_2 A_5 T_3 T_4) = V_\alpha^a(T_1 T_2 A) G_{\alpha\beta}^{ab}(A) V_{\beta\gamma}^{bc}(AA_5 A) G_{\gamma\lambda}^{cd}(A) V_\lambda^d(AT_3 T_4) \quad (21)$$

Where the vertex of one on-shell and two off-shell gauge fields is the following

$$V_{\beta\gamma}^{bc}(AA_5 A) = -T_9 (2\pi\alpha')^2 f^{\beta\gamma\alpha_5} \left((k_5 - q)^b (\xi_5^c)^{\alpha_5} + (p - k_5)^c (\xi_5^b)^{\alpha_5} + \eta^{bc}(q - p) \cdot \xi_5^{\alpha_5} \right)$$

where p and q are the momentum of off-shell gauge fields (see fig.3). Replacing the above vertex and the other vertex from (20) in (21), one finds

$$A'_k(T_1 T_2 A_5 T_3 T_4) = -(\alpha'^2 T_9) \zeta_1^{\alpha_1} \zeta_2^{\alpha_2} \zeta_3^{\alpha_3} \zeta_4^{\alpha_4} f^{\alpha_1 \alpha_2 \alpha} f^{\alpha \beta \alpha_5} f^{\alpha_3 \alpha_4 \beta} \frac{1}{s_{12} s_{34}}$$

$$\begin{aligned}
& (2(k_1 \cdot \xi_5^{\alpha_5} + k_2 \cdot \xi_5^{\alpha_5})(k_1 - k_2) \cdot (k_3 - k_4)) \\
& - ((k_1 \cdot \xi_5^{\alpha_5} - k_2 \cdot \xi_5^{\alpha_5})(k_1 + k_2 - k_5) \cdot (k_3 - k_4)) \\
& + ((k_3 \cdot \xi_5^{\alpha_5} - k_4 \cdot \xi_5^{\alpha_5})(k_1 - k_2) \cdot (k_3 + k_4 - k_5))
\end{aligned}$$

In the last line above, one should again use $k_4 \cdot \xi_5 = -(k_1 + k_2 + k_3) \cdot \xi_5$. To compare with the specific terms that we had chosen in string theory side, we consider only the coefficients $k_1 \cdot \xi_5$, *i.e.*,

$$\begin{aligned}
A'_k(T_1 T_2 A_5 T_3 T_4) &= -(\alpha' T_9) \zeta_1^{\alpha_1} \zeta_2^{\alpha_2} \zeta_3^{\alpha_3} \zeta_4^{\alpha_4} f^{\alpha_1 \alpha_2 \alpha} f^{\alpha \alpha_5 \beta} f^{\beta \alpha_3 \alpha_4} k_1 \cdot \xi_5^{\alpha_5} \\
&\times \left(\frac{1}{s_{34}} + \frac{1}{s_{12}} + \frac{4s_{23}}{s_{12}s_{34}} + \frac{4}{s_{12}s_{34}} \right) + \dots
\end{aligned}$$

where dots refer to the terms that have coefficients $k_2 \cdot \xi_5$ and $k_3 \cdot \xi_5$. The color factor in the above equation can be expressed as the following

$$-i \zeta_1^{\alpha_1} \zeta_2^{\alpha_2} \zeta_3^{\alpha_3} \zeta_4^{\alpha_4} f^{\alpha_1 \alpha_2 \alpha} f^{\alpha \alpha_5 \beta} f^{\beta \alpha_3 \alpha_4} k_1 \cdot \xi_5^{\alpha_5} = k_1 \cdot \xi_5 \text{Tr} \left([\lambda^1, \lambda^2] [\lambda^5, [\lambda^3, \lambda^4]] \right)$$

Obviously the above factor includes the ordering $-\text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5)$. By different labelling the diagram (b), one finds no other distinct diagram that produces the ordering $\text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5)$.

Finally, we consider the diagram (c) in fig1. The amplitude is given as

$$A''_k(A_5 T_3 T_4 T_1 T_2) = V_\alpha^a(A_5 T_3 T_4 A) G_{\alpha\beta}^{ab}(A) V_\beta^b(AT_1 T_2) \quad (22)$$

The vertex of two tachyons and two gauge fields in which only one of the gauge fields is off-shell is given by

$$V_\alpha^a(A_5 T_3 T_4 A) = -iT_9(2\pi\alpha') \zeta_3^{\alpha_3} \zeta_4^{\alpha_4} (\xi_5^a)^{\alpha_5} (f^{\alpha_5 \alpha_3 \rho} f^{\rho \alpha \alpha_4} + f^{\alpha_5 \alpha_4 \rho} f^{\rho \alpha \alpha_3})$$

After replacing the gauge field propagator and vertices in (22), one finds

$$\begin{aligned}
A''_k(A_5 T_3 T_4 T_1 T_2) &= -(i\alpha' T_9) \zeta_1^{\alpha_1} \zeta_2^{\alpha_2} \zeta_3^{\alpha_3} \zeta_4^{\alpha_4} (f^{\alpha_5 \alpha_3 \rho} f^{\rho \alpha \alpha_4} + f^{\alpha_5 \alpha_4 \rho} f^{\rho \alpha \alpha_3}) i f^{\alpha \alpha_1 \alpha_2} \\
&\times (k_1 \cdot \xi_5^{\alpha_5} - k_2 \cdot \xi_5^{\alpha_5}) \frac{1}{s_{12}}
\end{aligned}$$

The color factor can be written as

$$\begin{aligned}
& \zeta_1^{\alpha_1} \zeta_2^{\alpha_2} \zeta_3^{\alpha_3} \zeta_4^{\alpha_4} (f^{\alpha_5 \alpha_3 \rho} f^{\rho \alpha \alpha_4} + f^{\alpha_5 \alpha_4 \rho} f^{\rho \alpha \alpha_3}) i f^{\alpha \alpha_1 \alpha_2} \\
&= \zeta_1^{\alpha_1} \zeta_2^{\alpha_2} \left(\text{Tr} \left([\Lambda^{\alpha_5}, \lambda^3] [\Lambda^\alpha, \lambda^4] \right) + \text{Tr} \left([\Lambda^{\alpha_5}, \lambda^4] [\Lambda^\alpha, \lambda^3] \right) \right) i f^{\alpha \alpha_1 \alpha_2} \\
&= \text{Tr}([\lambda^5, \lambda^3] [\lambda^1, \lambda^2], \lambda^4] + \text{Tr}([\lambda^5, \lambda^4] [\lambda^1, \lambda^2], \lambda^3]
\end{aligned}$$

The final result will be

$$A''_k(A_5 T_3 T_4 T_1 T_2) = (i\alpha' T_9) \left(\text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5) + \dots \right) (k_1 \cdot \xi_5 - k_2 \cdot \xi_5) \frac{1}{s_{12}}$$

where dots refer to other orderings. By changing the labels in diagram (c), one finds two other distinct diagrams that produce the ordering $\text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5)$. They are

$$\begin{aligned} A_k''(A_5 T_1 T_2 T_3 T_4) &= (i\alpha' T_9)(k_3 \cdot \xi_5 - k_4 \cdot \xi_5) \frac{1}{s_{34}} \left(\text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5) + \dots \right) \\ A_k''(A_5 T_1 T_4 T_2 T_3) &= -(i\alpha' T_p)(k_2 \cdot \xi_5 - k_3 \cdot \xi_5) \frac{1}{s_{23}} \left(\text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5) + \dots \right) \end{aligned}$$

The second amplitude does not have $k_1 \cdot \xi_5$ contribution, however, using conservation of momentum, the term in the first amplitude above which is proportional to $k_4 \cdot \xi_5$ has $k_1 \cdot \xi_5$ contribution. The total terms in fig.1 which has coefficients $\text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5)$ and $k_1 \cdot \xi_5$ are then

$$\begin{aligned} \mathcal{A}_k^{TTTTA} &= (i\alpha' T_9) k_1 \cdot \xi_5 \left(\frac{4}{s_{45} + \frac{1}{2}} + \frac{4}{s_{15} + \frac{1}{2}} - \frac{4}{s_{23}} + \frac{4s_{23}}{(s_{45} + \frac{1}{2})s_{12}} + \frac{4s_{23}}{(s_{15} + \frac{1}{2})s_{34}} \right. \\ &\quad + \frac{4s_{34}}{(s_{15} + \frac{1}{2})s_{23}} + \frac{4s_{12}}{(s_{45} + \frac{1}{2})s_{23}} + \frac{4s_{23}}{s_{34}s_{12}} + \frac{4}{(s_{45} + \frac{1}{2})s_{12}} + \frac{4}{(s_{15} + \frac{1}{2})s_{34}} \\ &\quad \left. + \frac{4}{(s_{15} + \frac{1}{2})s_{23}} + \frac{4}{(s_{45} + \frac{1}{2})s_{23}} + \frac{4}{s_{34}s_{12}} \right) \text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5) + \dots \end{aligned} \quad (23)$$

This is exactly the leading order terms of the non-abelian expansion of the S-matrix element in the string theory side (13). This confirms that our prescription for expanding the string theory S-matrix element produces correctly the non-abelian expansion, an expansion whose first leading order terms are reproduced by the non-abelian kinetic terms. The next leading order terms in (13) are proportional to $\zeta(2)$. We now turn to the Feynman amplitudes in field theory that are proportional to $\zeta(2)$.

4.2 $\zeta(2)$ order terms

There are two main Feynman diagrams that are produced by one vertex from the terms in the second or third line of (17) and one vertex from the kinetic terms. They appear in fig.4. We write the amplitude as

$$\mathcal{A}_{\zeta(2)}^{TTTTA} = A_{\zeta(2)} + A'_{\zeta(2)}$$

where $A_{\zeta(2)}$ corresponds to diagram (a) and $A'_{\zeta(2)}$ corresponds to diagram (b). The Feynman amplitude of diagram (a) is given by

$$A_{\zeta(2)}(T_1 T_2 T_3 T_4 A_5) = V_\alpha(T_1 T_2 T_3 T) G_{\alpha\beta}(T) V_\beta(T T_4 A_5) \quad (24)$$

The vertex of four tachyons should be read from the different terms in the second line of (17). To do this, one should first perform the symmetric trace, *i.e.*,

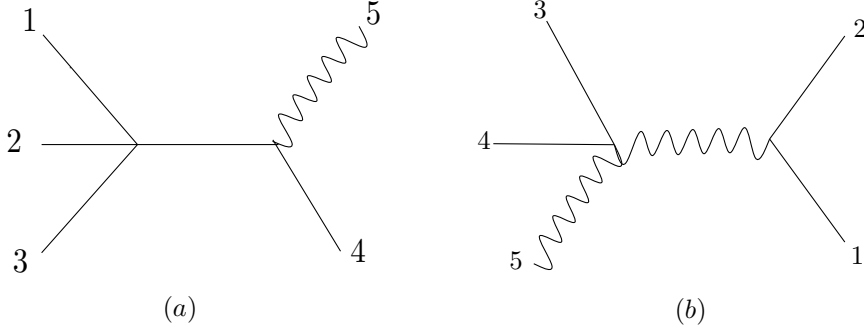


Figure 4: The Feynman diagrams at the $\zeta(2)$ order.

$$\begin{aligned}
\mathcal{L}^{TTTT} = & -T_9(2\pi\alpha')^2 \text{Tr} \left(\frac{m^4}{8} T^4 + \frac{m^2}{4} \left(\frac{2}{3} TT \partial_a T \partial^a T + \frac{1}{3} T \partial_a T T \partial^a T \right) \right. \\
& \left. - \frac{1}{24} \left(2 \partial_a T \partial^a T \partial_b T \partial^b T + \partial_a T \partial_b T \partial^a T \partial^b T \right) \right) \quad (25)
\end{aligned}$$

Now we can read the vertex of three on-shell tachyons and one off-shell tachyon from the above couplings. In this section, we choose to write the vertices in term of trace of λ 's. So the vertex of four tachyons contains terms that have group factors $\text{Tr}(\lambda^1 \lambda^2 \lambda^3 \Lambda^\alpha)$, $\text{Tr}(\lambda^2 \lambda^1 \lambda^3 \Lambda^\alpha)$, $\text{Tr}(\lambda^2 \lambda^3 \lambda^1 \Lambda^\alpha)$, $\text{Tr}(\lambda^3 \lambda^2 \lambda^1 \Lambda^\alpha)$, $\text{Tr}(\lambda^1 \lambda^3 \lambda^2 \Lambda^\alpha)$, and $\text{Tr}(\lambda^3 \lambda^1 \lambda^2 \Lambda^\alpha)$. However, after replacing them in (24), only the first factor will produce the desired ordering $\text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5)$. So, we consider only the terms in the vertex that have factor $\text{Tr}(\lambda^1 \lambda^2 \lambda^3 \Lambda^\alpha)$, *i.e.*,

$$\begin{aligned}
V_\alpha(T_1 T_2 T_3 T) = & T_9 i (2\pi\alpha')^2 \text{Tr}(\lambda^1 \lambda^2 \lambda^3 \Lambda^\alpha) \\
& \times \left(\frac{m^4}{2} - \frac{m^2}{6} (-p^2 + k_1 \cdot k_2 + k_2 \cdot k_3 + k_1 \cdot k_3) \right. \\
& \left. - \frac{1}{6} (k_1 \cdot k_2 k_3 \cdot p + k_2 \cdot k_3 k_1 \cdot p + k_1 \cdot k_3 k_2 \cdot p) \right) + \dots
\end{aligned}$$

where p is momentum of off-shell tachyon (see fig.5). Replacing this and the other vertex and propagator in (24), one finds

$$\begin{aligned}
A_{\zeta(2)}(T_1 T_2 T_3 T_4 A_5) = & -i\alpha' (8T_9 \pi^2) k_4 \cdot \xi_5^{\alpha_5} \left(\frac{1}{s_{45} + \frac{1}{2}} \right) \text{Tr}(\lambda^1 \lambda^2 \lambda^3 \Lambda^\alpha) i f^{\alpha\alpha_4\alpha_5} \zeta_4^{\alpha_4} \\
& \times \left(\alpha'^2 \frac{m^4}{2} - \alpha'^2 \frac{m^2}{6} (-p^2 + k_1 \cdot k_2 + k_2 \cdot k_3 + k_1 \cdot k_3) \right. \\
& - \frac{1}{6} [k_1 \cdot k_2 (k_3 \cdot p + k_3 \cdot k_5) + k_2 \cdot k_3 (k_1 \cdot p + k_1 \cdot k_5) \\
& \left. + k_1 \cdot k_3 (k_2 \cdot p + k_2 \cdot k_5)] \right) + \dots
\end{aligned}$$

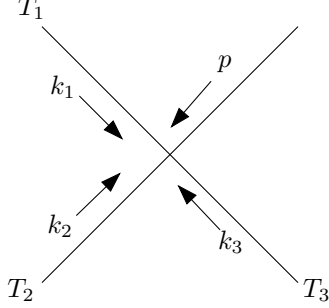


Figure 5: Vertex of three on-shell tachyons and one off-shell tachyon.

Keeping the ordering $\text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5)$ and factor $k_1 \cdot \xi_5$, one finds

$$\begin{aligned}
A_{\zeta(2)}(T_1 T_2 T_3 T_4 A_5) &= i\alpha' T_9 \zeta(2) \text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5) k_1 \cdot \xi_5 \\
&\times \left(-\frac{4s_{12}^2}{s_{45} + \frac{1}{2}} - \frac{4s_{12}s_{23}}{s_{45} + \frac{1}{2}} - \frac{4s_{23}^2}{s_{45} + \frac{1}{2}} - \frac{8s_{12}}{s_{45} + \frac{1}{2}} - \frac{8s_{23}}{s_{45} + \frac{1}{2}} - \frac{4}{s_{45} + \frac{1}{2}} \right. \\
&\quad \left. + 4s_{12} + 4s_{23} + 8 \right) + \dots
\end{aligned} \tag{26}$$

where we have used relation (6) and $\pi^2/6 = \zeta(2)$.

The other distinct diagram that produces the ordering $\text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5)$ is

$$\begin{aligned}
A_{\zeta(2)}(T_2 T_3 T_4 T_1 A_5) &= -i\alpha' T_9 \zeta(2) \text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5) (-k_1 \cdot \xi_5) \\
&\left(-\frac{4s_{23}^2}{s_{15} + \frac{1}{2}} - \frac{4s_{34}s_{23}}{s_{15} + \frac{1}{2}} - \frac{4s_{34}^2}{s_{15} + \frac{1}{2}} - \frac{8s_{23}}{s_{15} + \frac{1}{2}} - \frac{8s_{34}}{s_{15} + \frac{1}{2}} - \frac{4}{s_{15} + \frac{1}{2}} \right. \\
&\quad \left. + 4s_{23} + 4s_{34} + 8 \right) + \dots
\end{aligned} \tag{27}$$

where again dots refer to the terms that have coefficient $k_2 \cdot \xi_5$, $k_3 \cdot \xi_5$, and have group factor other than $\text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5)$.

We now consider the diagram (b) in fig. 4. The amplitude is given by

$$A'_{\zeta(2)}(A_5 T_1 T_2 T_3 T_4) = V_\alpha^a(A_5 T_1 T_2 A) G_{\alpha\beta}^{ab}(A) V_\beta^b(AT_3 T_4) \tag{28}$$

where the vertex of two tachyons and two gauge fields should be read from the terms in the third line of (17). Performing the symmetric trace, they can be written as

$$\mathcal{L}^{AATT} = -T_9 \frac{(2\pi\alpha')^3}{2} \text{Tr} \left(\left(\frac{1}{3} (\partial^a T \partial_b T F^{bc} F_{ca}) + \frac{1}{3} (\partial^a T \partial_b T F^{ca} F_{bc}) \right) \right)$$

$$\begin{aligned}
& +\frac{1}{6}(\partial^a T F^{bc} \partial_b T F_{ca}) + \frac{1}{6}(\partial^a T F^{ca} \partial_b T F_{bc}) \\
& -\frac{1}{4}\left(\frac{2}{3}(\partial^a T \partial_a T F^{bc} F_{cb}) + \frac{1}{3}(\partial^a T F^{bc} \partial_a T F_{cb})\right) \\
& -\frac{m^2}{4}\left(\frac{2}{3}(T T F^{ab} F_{ba}) + \frac{1}{3}(T F^{ab} T F_{ba})\right) \\
& \equiv \mathcal{L}_1^{AATT} + \mathcal{L}_2^{AATT} + \mathcal{L}_3^{AATT} + \mathcal{L}_4^{AATT} + \mathcal{L}_5^{AATT} + \mathcal{L}_6^{AATT} + \mathcal{L}_7^{AATT} + \mathcal{L}_8^{AATT}
\end{aligned} \tag{29}$$

The possible terms in the above interaction that produce in the amplitude (28) the ordering $\text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5)$ are $\mathcal{L}_1^{AATT} + \mathcal{L}_2^{AATT} + \mathcal{L}_5^{AATT} + \mathcal{L}_7^{AATT}$. One can find the vertex of two tachyons and two gauge fields from these terms. It contains terms that have group factors $\text{Tr}(\lambda^5 \lambda^1 \lambda^2 \Lambda^\alpha)$, $\text{Tr}(\lambda^1 \lambda^2 \lambda^5 \Lambda^\alpha)$, $\text{Tr}(\lambda^2 \lambda^1 \lambda^5 \Lambda^\alpha)$, and $\text{Tr}(\lambda^5 \lambda^2 \lambda^1 \Lambda^\alpha)$. However, after replacing them in (28), only the first factor will produce the desired ordering $\text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5)$. So, we consider only the terms in the vertex that have factor $\text{Tr}(\lambda^5 \lambda^1 \lambda^2 \Lambda^\alpha)$. By replacing it and the other vertex from kinetic term in (28), one finds that the amplitude has terms proportional to $k_1 \cdot \xi_5$, $k_2 \cdot \xi_5$, $k_3 \cdot \xi_5$, and $k_4 \cdot \xi_5$. The last factor should be written as $-(k_1 + k_2 + k_3) \cdot \xi_5$. To compare the result with the terms in the string theory side (13), we consider only terms that have coefficient $k_1 \cdot \xi_5$ and group factor $\text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5)$. The result is

$$\begin{aligned}
A'_{\zeta(2)}(A_5 T_1 T_2 T_3 T_4) &= i\alpha' T_9 \zeta(2) k_1 \cdot \xi_5 \text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5) \\
&\times \left(-\frac{4s_{23}(s_{15} + \frac{1}{2})}{s_{34}} - \frac{4s_{12}s_{23}}{s_{34}} - \frac{4(s_{15} + \frac{1}{2})}{s_{34}} - \frac{4s_{12}}{s_{34}} \right. \\
&\quad \left. - 2(s_{15} + \frac{1}{2}) - 2s_{12} + 2s_{34} + 4s_{23} + 4 \right) + \dots
\end{aligned} \tag{30}$$

Another distinct diagram that produces the ordering $\text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5)$ and has coefficient $k_1 \cdot \xi_5$ is

$$\begin{aligned}
A'_{\zeta(2)}(T_3 T_4 A_5 T_1 T_2) &= V_\alpha^a(T_3 T_4 A_5 A) G_{\alpha\beta}^{ab}(A) V_\beta^b(AT_1 T_2) \\
&= i\alpha' T_9 \zeta(2) k_1 \cdot \xi_5 \text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5) \\
&\times \left(-\frac{4s_{23}(s_{45} + \frac{1}{2})}{s_{12}} - \frac{4s_{23}s_{34}}{s_{12}} - \frac{4(s_{45} + \frac{1}{2})}{s_{12}} - \frac{4s_{34}}{s_{12}} \right. \\
&\quad \left. - 2(s_{45} + \frac{1}{2}) + 2s_{12} + 4s_{23} - 2s_{34} + 4 \right) + \dots
\end{aligned} \tag{31}$$

In above amplitude again only $\mathcal{L}_1^{AATT} + \mathcal{L}_2^{AATT} + \mathcal{L}_5^{AATT} + \mathcal{L}_7^{AATT}$ has contribution. The remaining terms in (29), *i.e.*, $\mathcal{L}_3^{AATT} + \mathcal{L}_4^{AATT} + \mathcal{L}_6^{AATT} + \mathcal{L}_8^{AATT}$ has contribution into the following amplitude:

$$\begin{aligned}
A'_{\zeta(2)}(T_4 A_5 T_1 T_2 T_3) &= V_\alpha^a(T_4 A_5 T_1 A) G_{\alpha\beta}^{ab}(A) V_\beta^b(AT_2 T_3) \\
&= i\alpha' T_9 \zeta(2) k_1 \cdot \xi_5 \text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5)
\end{aligned}$$

$$\begin{aligned} & \times \left(\frac{4(s_{15} + \frac{1}{2})(s_{45} + \frac{1}{2})}{s_{23}} - \frac{4(s_{15} + \frac{1}{2})s_{12}}{s_{23}} - \frac{4(s_{45} + \frac{1}{2})s_{34}}{s_{23}} \right. \\ & \left. - \frac{4(s_{15} + \frac{1}{2})}{s_{23}} - \frac{4(s_{45} + \frac{1}{2})}{s_{23}} - 2(s_{15} + \frac{1}{2}) - 2(s_{45} + \frac{1}{2}) \right) + \dots \end{aligned} \quad (32)$$

Comparing terms in $\mathcal{A}_{\zeta(2)}^{TTTTA}$ which have coefficients $\text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5)$ and $k_1 \cdot \xi_5$, *i.e.*, equations (26), (27), (30), (31), and (32), with the non-abelian expansion of string theory S-matrix element (13), one finds that the poles of (13) which are proportional to $\zeta(2)$ are exactly reproduced by the above field theory amplitudes. This confirms that while the leading order terms of the non-abelian expansion of S-matrix element (13) are reproduced by non-abelian kinetic terms (23), the next leading order terms of the S-matrix element are reproduced by non-abelian tachyon DBI action. In particular, it confirms the four-tachyon couplings in (25). These terms appear in one of the vertex in diagram (a) in fig.4. Since one of the legs of the vertex is off-shell, there is no on-shell ambiguity in these terms. There is such ambiguity in finding them from studying the non-abelian expansion of the S-matrix element of four tachyons [45]. In that calculation, the couplings (25) appear as contact term in field theory, hence, there is on-shell ambiguity⁶.

We have seen that the tachyon and massless poles in the non-abelian expansion of the string theory S-matrix element (13) are exactly reproduced by the tachyon and massless poles of the non-abelian DBI action (14). One can even show that the contact terms of (13) which appear in the next leading order terms, *i.e.*, the contact terms which are proportional to $\zeta(2)$, can be reproduced by non-abelian DBI action. To this end, we subtract the field theory amplitudes $\mathcal{A}_k^{TTTTA} + \mathcal{A}_{\zeta(2)}^{TTTTA}$ from (13), *i.e.*,

$$\mathcal{A}^{TTTTA} - \mathcal{A}_k^{TTTTA} - \mathcal{A}_{\zeta(2)}^{TTTTA} = i\alpha' T_9 \zeta(2) k_1 \cdot \xi_5 \text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5) \left(4s_{23} + 8 \right) + \dots$$

where dots refer to the terms with coefficients $\zeta(3)$, $\zeta(4)$, and so on. One can easily check that the above contact terms are exactly the four tachyons and one gauge field couplings in the second line of (17). This ends our illustration of consistency between the non-abelian expansion of S-matrix element (13) and the non-abelian DBI action (14).

The next order terms in the expansion (13) are proportional to $\zeta(3)$. They contains massless, tachyon poles, and contact terms. Like the terms in the kinetic order and $\zeta(2)$ order, the different terms of $\zeta(3)$ -order are not of the same α' order. We speculate that the $\zeta(3)$ -order terms in (13) are reproduced by second derivative of T , *i.e.*, couplings that include $\partial\partial T$. Similarly, in studying the non-abelian expansion of S-matrix element of four tachyons [45], the kinetic order terms and $\zeta(2)$ -order terms are shown to be reproduced by tachyon DBI action. In that case also the next leading order terms which are only contact terms are proportional to $\zeta(3)$. Those terms may also be written as second derivative of T , *e.g.*, $\alpha'^2 T^2 \partial_a \partial_b T \partial^a \partial^b T$ or $\alpha'^3 \partial_a T \partial^a T \partial_b \partial_c T \partial^b \partial^c T$. However, there are on-shell ambiguity in

⁶See [60], for an off-shell T^4 coupling in cubic string field theory.

these couplings, *e.g.*, $\alpha'^2 T^2 \partial_a \partial_b T \partial^a \partial^b T \sim \alpha'^3 T \partial_c \partial^c T \partial_a \partial_b T \partial^a \partial^b T$. This ambiguity can be fixed by studying them in the S-matrix element of four tachyons and one gauge field in which the $\zeta(3)$ -order terms have tachyon poles. It would be interesting to explicitly perform these calculations to find the second-derivative corrections to the tachyon DBI action. Another interesting calculation is to evaluate the S-matrix element of six tachyons. Comparing the result with the S-matrix element of six massless scalars [61], one would find the non-abelian expansion of the tachyon amplitude. This calculation would fix the T^6 term in the tachyon potential (15).

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A appendix

The momentum dependent factors a', b', c' in the tachyon scattering amplitude (3), and $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ in the scalar scattering amplitude (10) are the double integral that left over after fixing position of three vertex operators in (1) and (9), respectively. In general, using Mathematica software, one can write the following double integral in terms of two Beta functions and one hypergeometric function:

$$\begin{aligned} & \int_0^1 dx_3 \int_0^{x_3} dx_2 x_3^{-s_{13}-n_{13}} (1-x_3)^{-s_{34}-n_{34}} x_2^{-s_{12}-n_{12}} (1-x_2)^{-s_{24}-n_{24}} (x_3-x_2)^{-s_{23}-n_{23}} \\ &= \beta(1-s_{12}-n_{12}, 1-s_{23}-n_{23}) \beta(2-s_{12}-s_{13}-s_{23}-n_{12}-n_{13}-n_{23}, 1-s_{34}-n_{34}) \\ & \quad \times {}_3F_2(s_{24}+n_{24}, 1-s_{12}-n_{12}, 2-s_{12}-s_{13}-s_{23}-n_{12}-n_{13}-n_{23}; \\ & \quad 2-s_{12}-s_{23}-n_{12}-n_{23}, 3-s_{34}-s_{12}-s_{13}-s_{23}-n_{34}-n_{12}-n_{13}-n_{23}; 1) \end{aligned}$$

The integers n_{ij} for each factor are listed in the following table:

momentum factors	n_{24}	n_{23}	n_{12}	n_{34}	n_{13}
a_1	0	1	1	1	0
a_2	1	1	0	0	1
a_3	0	2	0	0	0
b_1	0	0	1	1	1
b_2	1	0	0	0	2
b_3	0	1	0	0	1
c_1	0	0	2	1	0
c_2	1	0	1	0	1
c_3	0	1	1	0	1
a'	1	1	2	1	1
b'	1	2	1	1	1
c'	2	1	1	1	1

The factors $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ are the same factors that appear in the S-matrix element of five gauge fields [56]. There, they have been named $L_2, L_7, K_6, L_3, L_6, K_5, L_5, L_1, K_4$, respectively .

The factor in the tachyon amplitude (3) which is multiplied by the kinematic factor $k_1 \cdot \xi_5$ in which we are interested is

$$\begin{aligned} a' &= \beta(-s_{12}, -s_{23}-1) \beta(-s_{45} - \frac{1}{2}, -s_{34}) \\ & \times {}_3F_2(-s_{12}, -s_{45} - \frac{1}{2}, s_{15} - s_{23} - s_{34} - \frac{1}{2}; -s_{12} - s_{23} - 1, -s_{34} - s_{45} - \frac{1}{2}; 1) \end{aligned}$$

We are going to find the non-abelian expansion of this factor. Consider the following function:

$$a = \beta(l_1, l_2)\beta(l_3, l_4) {}_3F_2(m_1, m_2, m_3; n_1, n_2; 1)$$

If one chooses appropriate value for l_i , m_i , and n_i , the above function converts to a' or to one of the other momentum factors. For the non-abelian expansion of a' , one should expand the above function around three different points (11). Expansion of Beta function is straightforward, using Mathematica,

$$\begin{aligned} \lim_{l_1, l_2, l_3, l_4 \rightarrow 0} \beta(l_1, l_2)\beta(l_3, l_4) &= \frac{(l_1 + l_2)(l_3 + l_4)}{l_1 l_2 l_3 l_4} - \zeta(2) \frac{(l_1 + l_2)(l_3 + l_4)(l_1 l_2 + l_3 l_4)}{l_1 l_2 l_3 l_4} + \dots \\ \lim_{l_1, l_4 \rightarrow 1, l_2, l_3 \rightarrow 0} \beta(l_1, l_2)\beta(l_3, l_4) &= \frac{(1 + l_1 + l_2)(1 + l_3 + l_4)}{(1 + l_1)l_2 l_3(1 + l_4)} \\ &\quad - \zeta(2) \frac{(1 + l_1 + l_2)(1 + l_3 + l_4)(l_2 + l_3 + l_1 l_2 + l_3 l_4)}{(1 + l_1)l_2 l_3(1 + l_4)} + \dots \\ \lim_{l_1, l_4 \rightarrow 1, l_2 \rightarrow -1, l_3 \rightarrow 0} \beta(l_1, l_2)\beta(l_3, l_4) &= \frac{(l_1 + l_2)(1 + l_3 + l_4)}{(1 + l_1)(-1 + l_2)l_3(1 + l_4)} \\ &\quad - \zeta(2) \frac{(l_1 + l_2)(1 + l_3 + l_4)(-1 + l_2 - l_1 + l_3 + l_1 l_2 + l_3 l_4)}{(1 + l_1)(-1 + l_2)l_3(1 + l_4)} \\ &\quad + \dots \end{aligned} \tag{33}$$

where dots refer to the terms that have coefficients $\zeta(3)$, $\zeta(4)$ and so on. The nontrivial part of expansion is the expansion of Hypergeometric function. This function appears also in higher terms of the ε -expansion of massive Feynman diagrams [57]. There are different approaches for expanding this function [51, 58]. In these references, one can find many useful techniques for expansion explicitly the Hypergeometric function. Fortunately, there is a package for expanding the Hypergeometric functions [59]. Using this package, one finds the following expansion for Hypergeometric function around the three point (11):

$$\begin{aligned} \lim_{m_1, m_2, m_3, n_1, n_2 \rightarrow 0} {}_3F_2(m_1, m_2, m_3; n_1, n_2; 1) &= \frac{n_1 n_2 (-m_2 - m_3 + n_1 + n_2) + m_1 (m_2 m_3 - n_1 n_2)}{n_1 n_2 (-m_1 - m_2 - m_3 + n_1 + n_2)} \\ &\quad - \zeta(2) \frac{m_1 m_2 m_3}{n_1 n_2 (m_1 + m_2 + m_3 - n_1 - n_2)} (n_1^2 - m_3 n_1 + n_2 n_1 + n_2^2 \\ &\quad + m_2 (m_3 - n_1 - n_2) + m_1 (m_2 + m_3 - n_1 - n_2) - m_3 n_2) + \dots \\ \lim_{m_1, m_3, n_1, n_2 \rightarrow 1, m_2 \rightarrow 0} {}_3F_2(m_1, m_2, m_3; n_1, n_2; 1) &= \frac{1}{(n_1 + 1)(n_2 + 1)(n_1 + n_2 - m_1 - m_2 - m_3)} \\ &\quad \times (n_1 + n_2 + (n_1 + n_2 - m_2)(n_1 + n_2 + n_1 n_2) + m_3 (m_2 - (n_1 + 1)(n_2 + 1)) \\ &\quad + m_1 (m_2 (m_3 + 1) - (n_1 + 1)(n_2 + 1))) \\ &\quad + \zeta(2) \frac{m_2 (1 + m_1)(1 + m_3)}{(n_1 + 1)(n_2 + 1)(n_1 + n_2 - m_1 - m_2 - m_3)} (n_2^2 + n_2 (1 + n_1 - m_2 - m_3)) \end{aligned}$$

$$\begin{aligned}
& +(m_2 - n_1 - 1)(m_3 - n_1) + m_1(m_2 + m_3 - n_1 - n_2 - 1)) + \dots \\
& \lim_{m_1, n_2 \rightarrow 1, m_2, m_3, n_1 \rightarrow 0} {}_3F_2(m_1, m_2, m_3; n_1, n_2; 1) = \frac{1}{n_1(1 + n_2)(n_1 + n_2 - m_1 - m_2 - m_3)} \\
& \times (n_1(1 + n_2)(n_1 + n_2 - m_1 - m_3) + m_2(m_3(1 + m_1) - n_1(1 + n_2))) \\
& + \zeta(2) \frac{m_2 m_3(1 + m_1)}{n_1(1 + n_2)(m_1 + m_2 + m_3 - n_1 - n_2)} (n_1^2 + n_2^2 + n_1 n_2 - n_1 m_3 \\
& + m_2(m_3 - n_1 - n_2) + m_1(m_2 + m_3 - n_1 - n_2 - 1) - m_3 n_2 + n_2) + \dots
\end{aligned} \tag{34}$$

Using the expansions (33) and (34), and choosing the specific value of l_i , m_i and n_i corresponding to the momentum factor a' , one finds the following non-abelian expansion:

$$\begin{aligned}
a'_1 &= \lim_{s_{23} \rightarrow -1, s_{45}, s_{15} \rightarrow -\frac{1}{2}, s_{12}, s_{34} \rightarrow 0} a' = \frac{1}{s_{12}(s_{45} + \frac{1}{2})} + \frac{1}{s_{34}(s_{15} + \frac{1}{2})} \\
& + \frac{1}{s_{12}s_{34}} + \frac{1}{(s_{23} + 1)(s_{45} + \frac{1}{2})} + \frac{1}{(s_{23} + 1)(s_{15} + \frac{1}{2})} \\
& - \zeta(2) \left(\frac{(s_{15} + \frac{1}{2})}{(s_{23} + 1)} + \frac{(s_{15} + \frac{1}{2})}{s_{34}} + \frac{(s_{45} + \frac{1}{2})}{(s_{23} + 1)} + \frac{s_{12}}{(s_{45} + \frac{1}{2})} \right. \\
& \left. + \frac{s_{34}}{s_{12}} + \frac{(s_{23} + 1)}{(s_{45} + \frac{1}{2})} + \frac{(s_{45} + \frac{1}{2})}{s_{12}} + \frac{s_{12}}{s_{34}} + \frac{(s_{23} + 1)}{(s_{15} + \frac{1}{2})} + \frac{s_{34}}{(s_{15} + \frac{1}{2})} - 3 \right) + \dots \\
a'_2 &= \lim_{s_{12}, s_{23}, s_{34} \rightarrow -1, s_{45}, s_{15} \rightarrow -\frac{1}{2}} a' = \frac{1}{(s_{15} + \frac{1}{2})(s_{23} + 1)} + \frac{1}{(s_{45} + \frac{1}{2})(s_{23} + 1)} \\
& - \zeta(2) \left(\frac{(s_{12} + 1)}{(s_{45} + \frac{1}{2})} + \frac{(s_{34} + 1)}{(s_{15} + \frac{1}{2})} + \frac{(s_{45} + \frac{1}{2})}{(s_{23} + 1)} + \frac{(s_{15} + \frac{1}{2})}{(s_{23} + 1)} - 1 \right) + \dots \\
a'_3 &= \lim_{s_{12}, s_{34} \rightarrow -1, s_{45}, s_{15} \rightarrow -\frac{1}{2}, s_{23} \rightarrow 0} a' = -\frac{(s_{12} + 1)}{(s_{45} + \frac{1}{2})} + \frac{(s_{12} + 1)}{s_{23}(s_{45} + \frac{1}{2})} - \frac{(s_{34} + 1)}{(s_{15} + \frac{1}{2})} \\
& - \frac{s_{23}}{(s_{15} + \frac{1}{2})} - \frac{s_{23}}{(s_{45} + \frac{1}{2})} + \frac{1}{(s_{45} + \frac{1}{2})} + \frac{1}{(s_{15} + \frac{1}{2})} + \frac{(s_{34} + 1)}{s_{23}(s_{15} + \frac{1}{2})} - \frac{1}{s_{23}} + 1 \\
& + \frac{s_{23}(1 + s_{12})}{(s_{45} + \frac{1}{2})} + \frac{s_{23}(s_{34} + 1)}{(s_{15} + \frac{1}{2})} - s_{23} + \frac{s_{23}^2}{(s_{15} + \frac{1}{2})} + \frac{s_{23}^2}{(s_{45} + \frac{1}{2})} + \zeta(2) \left(-\frac{(1 + s_{12})^2}{(s_{45} + \frac{1}{2})} \right. \\
& - \frac{s_{23}(1 + s_{12})}{(s_{45} + \frac{1}{2})} - \frac{(s_{15} + \frac{1}{2})(s_{12} + 1)}{s_{23}} - \frac{s_{23}(s_{34} + 1)}{(s_{15} + \frac{1}{2})} + \frac{(s_{15} + \frac{1}{2})(s_{45} + \frac{1}{2})}{s_{23}} \\
& \left. - \frac{(s_{34} + 1)(s_{45} + \frac{1}{2})}{s_{23}} - \frac{(s_{34} + 1)^2}{(s_{15} + \frac{1}{2})} + s_{12} + s_{23} - s_{45} - s_{15} + s_{34} + 1 \right) + \dots
\end{aligned}$$

Inserting the above expansion in (12), one finds the non-abelian expansion of the tachyon amplitude (13).

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